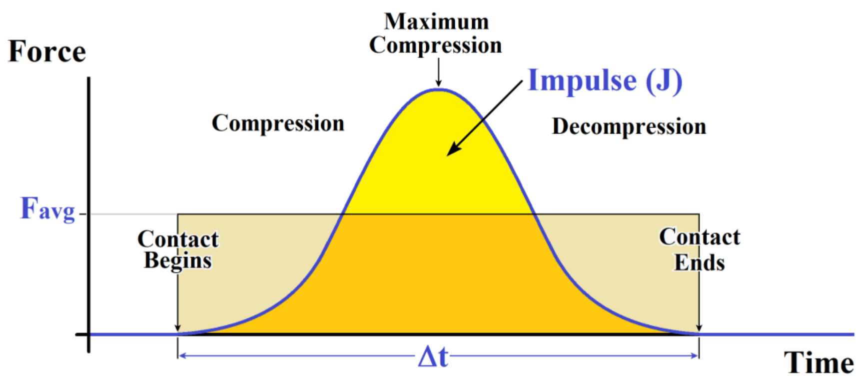
Part 8: Impulse, and Momentum

University Physics V1 (Openstax): Chapters 9
Physics for Engineers & Scientists (Giancoli): Chapters 8

Impacts and Impulse

- Collisions generally occur over a short time interval.
- When two objects collide, the surfaces will deform (surfaces compress).
- These surfaces act like springs (albeit with large spring constants). The greater the compression, the greater the force. This allows the contact forces (equivalent to normal forces) to slow the colliding objects and redirect them.
- Some of the collision energy may be released as heat (warming the colliding objects) and sound. Other energy might be consumed in permanently deforming one or both of the objects. As these energies are difficult to account for, conservation of energy can only be used special cases.



- **Impulse** ($\vec{J}$): $\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)dt = \vec{F}_{avg} \cdot \Delta t$

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)dt = \int_{t_1}^{t_2} m\vec{a}(t)dt = m\int_{t_1}^{t_2} \vec{a}(t)dt = m[\vec{v}(t)]_{t_1}^{t_2} = m\vec{v}_2 - m\vec{v}_1$$

- **Momentum** ($\vec{P}$): $\vec{P} = m\vec{v}$
- The impulse delivered to an object is equal to that object's change in momentum.

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)dt = \vec{F}_{avg} \cdot \Delta t = m\vec{v}_2 - m\vec{v}_1 = \Delta\vec{P}$$

- Which also means: $\vec{F}_{avg} = \frac{\Delta\vec{P}}{\Delta t}$ $\quad \vec{F} = \frac{d\vec{P}}{dt}$

Example: A golf ball (m_b = 45.0 g) starts at rest on a tee. After the golfer strikes it, it is moving at 38.0 m/s. During impact, the club remains in contact with the ball for 3.00 ms. A) What is the change in momentum of the ball? B) Determine the average force applied to the ball by the club.

A) $\Delta P = P_{Final} - P_{Init} = mv - mv_0 = m(v - v_0) = (0.0450\,kg)\left(38.0\frac{m}{s} - 0\frac{m}{s}\right) = 1.71\,kg \cdot \frac{m}{s}$

B) $F_{Avg} = \frac{\Delta P}{\Delta t} = \frac{1.71\,kg \cdot \frac{m}{s}}{3.00 \times 10^{-3}\,s} = 570\,N$

Example: The Space-X Falcon 9 rocket has a mass of 1.48×10^6 kg when loaded with payload destined for low-Earth orbit (leo). Its engines generate a force given by $F(t) = \beta(1-e^{-\sigma t})$ where $\beta = 2.28 \times 10^7$ N and $\sigma = 0.139\ s^{-1}$. If the rocket is at rest at t = 0, how fast is it moving at t = 5.00 s?

This problem is one dimensional (vertically upward). Vectors can be dispensed with.

$$J = \int_{t_1}^{t_2} F(t)dt = mv$$

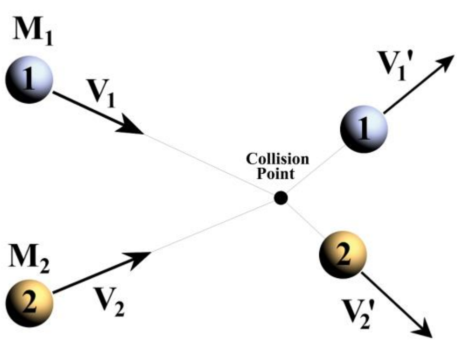
$$J = \int_{t_1}^{t_2} F(t)dt = \int_0^t [\beta(1-e^{-\sigma t})]dt = \int_0^t [\beta - \beta e^{-\sigma t}]dt = \int_0^t \beta dt + \int_0^t -\beta e^{-\sigma t} dt$$

$$J = [\beta t]_0^t - \left[-\frac{\beta}{\sigma} e^{-\sigma t}\right]_0^t = \beta t - \frac{\beta}{\sigma}(1-e^{-\sigma t}) = \beta\left[t - \frac{1}{\sigma}(1-e^{-\sigma t})\right]$$

$$J = (2.28 \times 10^7\ N)\left\{(5.00\ s) - \frac{1}{(0.139\ s^{-1})}\left[1 - e^{-(0.139\ s^{-1})(5.00\ s)}\right]\right\} = 3.1834 \times 10^7\ N \cdot s$$

$$v = \frac{J}{m} = \frac{3.1834 \times 10^7\ N \cdot s}{1.48 \times 10^6\ kg} = 21.5\frac{m}{s}$$

General Collision



Object 1: $\vec{J}_1 = \vec{F}_1 \cdot \Delta t = \Delta \vec{P}_1 = m\vec{v}_1' - m\vec{v}_1$

Object 2: $\vec{J}_2 = \vec{F}_2 \cdot \Delta t = \Delta \vec{P}_2 = m\vec{v}_2' - m\vec{v}_2$

$$\vec{J}_1 + \vec{J}_2 = \vec{F}_1 \cdot \Delta t + \vec{F}_2 \cdot \Delta t = (\vec{F}_1 + \vec{F}_2) \cdot \Delta t = 0$$

$\vec{F}_1 + \vec{F}_2 = 0$, *because Newton's 3rd law says* $\vec{F}_2 = -\vec{F}_1$

$$\vec{J}_1 + \vec{J}_2 = (m\vec{v}_1' - m\vec{v}_1) + (m\vec{v}_2' - m\vec{v}_2)$$

$$\vec{J}_1 + \vec{J}_2 = (m\vec{v}_1' + m\vec{v}_2') - (m\vec{v}_1 + m\vec{v}_2) = \vec{P}_{Final} - \vec{P}_{Init}$$

$$\vec{P}_{Final} - \vec{P}_{Init} = 0 \qquad \vec{P}_{Final} = \vec{P}_{Init}$$

Momentum is conserved!

...As long as no external forces provide outside impulse.

Example: A defensive lineman (m_{DL} = 138 kg) is moving at 8.00 m/s when he tackles a stationary quarterback (m_{QB} = 110 kg). A) What is the velocity of the pair after the collision? B) If the collision takes 0.200 s, what is the average force delivered to the quarterback?

A) $P_{Final} = P_{Init} \qquad (m_{QB} + m_{DL})v = m_{DL}v_0 \qquad v = \frac{m_{DL}v_0}{m_{QB}+m_{DL}} = \frac{(138\ kg)\left(8.00\frac{m}{s}\right)}{(110\ kg+138\ kg)} = 4.45\frac{m}{s}$

B) $F_{Avg} = \frac{\Delta P_{QB}}{\Delta t} = \frac{m_{QB}v - 0}{\Delta t} = \frac{(110\ kg)\left(4.45\frac{m}{s}\right)}{(0.200\ s)} = 2.45\ kN$ *(approximately 550 lbs of force)*

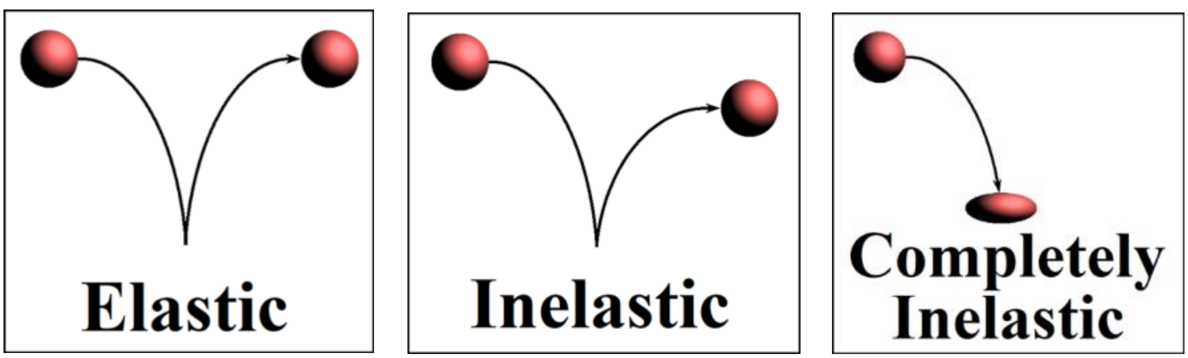
$$\Delta P_{QB} = m_{QB}v = (110\ kg)\left(4.45\frac{m}{s}\right) = 490\ kg \cdot \frac{m}{s}$$

$$\Delta P_{DL} = m_{DL}v - m_{DL}v_0 = m_{DL}(v - v_0) = (138\ kg)\left(4.45\frac{m}{s} - 8.00\frac{m}{s}\right) = -490\ kg \cdot \frac{m}{s}$$

The momentum lost by the defensive lineman is transferred to the quarterback

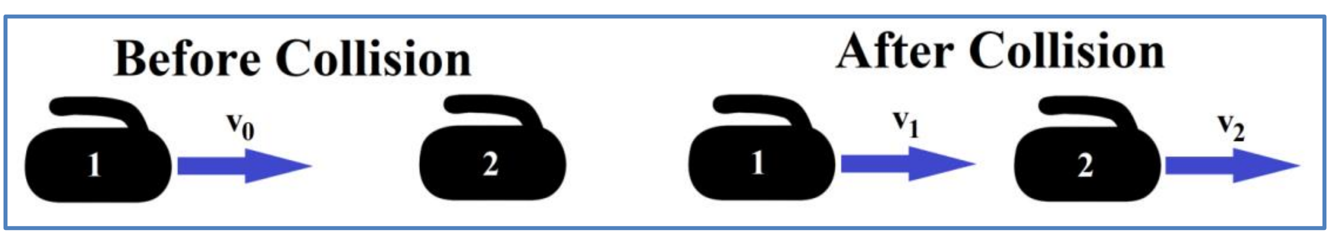
Energy and Momentum in Collisions

- Momentum is conserved in collisions when no net outside forces are present.
- Gravity is typically negligible as it has almost no effect over such a short time interval.
- Energy may or may not be conserved in collisions. For example, energy might be released as heat (lost).
- Three types of collisions:
  - In Elastic Collisions, both energy and momentum are conserved.
  - In Inelastic Collisions, momentum is conserved, but energy is not.
  - In Completely Inelastic Collisions, momentum is conserved, but the maximum possible energy is lost as the objects stick together.



Example: A curler slides a 20.0 kg stone across the ice surface. The stone is moving at 0.750 m/s when it collides head-on with a second, stationary 20.0 kg curling stone. If this is an elastic collision, determine the velocity of both stones after they strike.

Note: The term 'head on' indicates that the center of mass of both objects is in line with the velocity. In other words, this is a 1-dimensional problem. Vectors can be ignored.



Conservation of momentum: $P_{init} = P_{Final}$ $\quad m_1v_0 = m_1v_1 + m_2v_2$

Conservation of energy: $E_{init} = E_{Final}$ $\quad \frac{1}{2}m_1v_0^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$ $\quad m_1v_0^2 = m_1v_1^2 + m_2v_2^2$

Here's a mathematical trick that helps with elastic collisions:

Rearrange momentum equation: $m_1(v_0 - v_1) = m_2 v_2$

Rearrange and factor energy equation: $m_1(v_0^2 - v_1^2) = m_2 v_2^2$ $\quad m_1(v_0 - v_1)(v_0 + v_1) = m_2 v_2^2$

Divide equations: $\frac{m_1(v_0 - v_1)(v_0 + v_1)}{m_1(v_0 - v_1)} = \frac{m_2 v_2^2}{m_2 v_2}$ $\quad v_0 + v_1 = v_2$

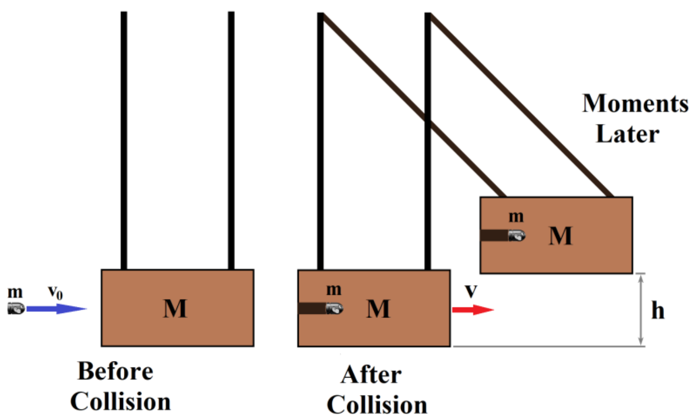
Multiply by m_1 and add the rearranged momentum equation:

$$m_1(v_0 + v_1) + m_1(v_0 - v_1) = m_1(v_2) + m_2(v_2) \qquad 2m_1 v_0 = (m_1 + m_2)v_2 \qquad v_2 = \frac{2m_1 v_0}{(m_1 + m_2)}$$

$$v_1 = v_2 - v_0 = \frac{2m_1 v_0}{(m_1 + m_2)} - \frac{(m_1 + m_2)v_0}{(m_1 + m_2)} = \frac{(m_1 - m_2)v_0}{(m_1 + m_2)}$$

As $m_1 = m_2$ in this case… $v_2 = v_0 = 0.75\frac{m}{s}$ and $v_1 = 0$

Ballistic Pendulum



- The ballistic pendulum is used to measure the velocity of projectiles (such as bullet).
- First, the projectile makes a completely inelastic collision with the much heavier hanging mass of a pendulum bob.
- The velocity of the pair after the collision causes the pendulum bob to swing upwards, and the height is measured. From this height we can produce the velocity of the projectile.

- To start we need to relate the 'Moments Later' image to the 'After Collision' image.
  - Can we use conservation of momentum? No, an external force (gravity) acts on the system. Momentum is not conserved.
  - There is no collision in this interval. Energy is conserved.

$$E_{Init} = E_{Final} \qquad \frac{1}{2}(m + M)v^2 = (m + M)gh \qquad v^2 = 2gh \qquad v = \sqrt{2gh}$$

- Next we need to relate the 'After Collision' image to the 'Before Collision' image.
  - Can we use conservation of energy? No, a collision occurs. Energy is not conserved.
  - There are no (horizontal) external forces in this interval. Momentum is conserved.

$$P_{Init} = P_{Final} \qquad mv_0 = (m + M)v \qquad v_0 = \left(1 + \frac{M}{m}\right)v = \left(1 + \frac{M}{m}\right)\sqrt{2gh}$$

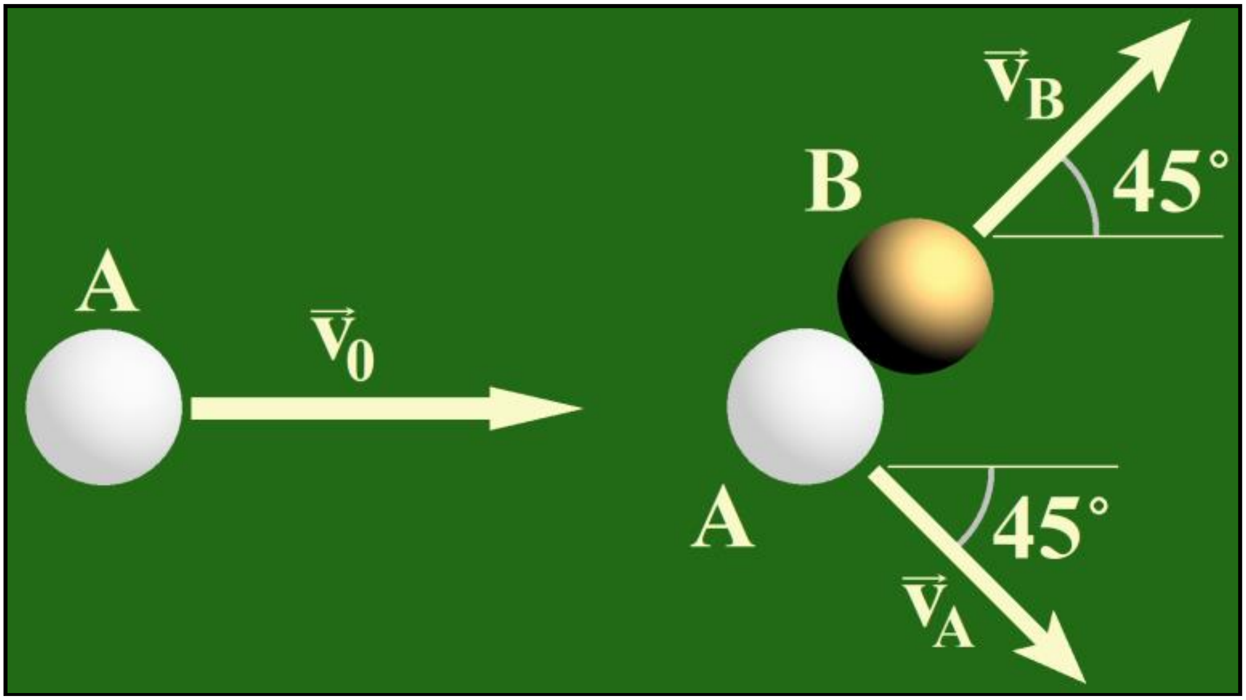
Collisions in Two Dimensions

- Conservation of momentum is applied by components for each axis of motion.

$$\vec{P}_{final} = \vec{P}_{init} \quad \text{implies} \quad P_{final-x} = P_{init-x} \quad \text{and} \quad P_{final-y} = P_{init-y}$$

- In two dimensions, conservation of motion gives two equations, allowing you to find two unknowns.

<u>Example</u>: The cue ball approaches a stationary ball of equal mass at 3.00 m/s. After the collision the balls separate, the velocity of each ball making a 45.0° angle with the cue ball's original path just on opposite sides. Determine the velocity of both balls after the collision.



<u>Y-Components</u>: $\qquad P_{final-y} = P_{init-y} \qquad m_B v_B \sin\theta_B - m_A v_A \sin\theta_A = 0$

$$m_B v_B \sin\theta_B = m_A v_A \sin\theta_A \qquad v_B = v_A$$

<u>Y-Components</u>: $\qquad P_{final-x} = P_{init-x} \qquad m_B v_B \cos\theta_B + m_A v_A \cos\theta_A = m_A v_0$

$$v_B \cos 45° + v_A \cos 45° = v_0 \qquad v_A \cos 45° + v_A \cos 45° = v_0$$

$$2v_A \cos 45° = v_0 \qquad v_A = \frac{v_0}{2\cos 45°} = \frac{3.00\frac{m}{s}}{2\cos 45°} = 2.12132\frac{m}{s} \qquad v_B = v_A = 2.12\frac{m}{s}$$

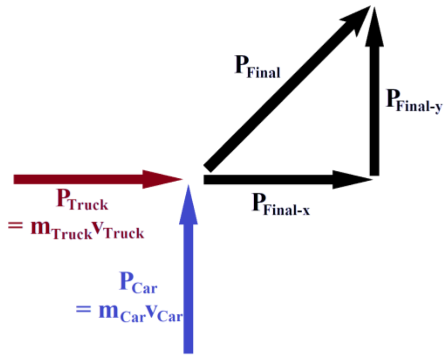
Is this an elastic or inelastic collision?

$$E_{init} = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(3.00\frac{m}{s}\right)^2 = \left(4.50\ \frac{m^2}{s^2}\right)m$$

$$E_{final} = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = mv_A^2 = m\left(2.12132\frac{m}{s}\right)^2 = \left(4.50\ \frac{m^2}{s^2}\right)m$$

So it's elastic.

Example: A car (m_{car} = 1300 kg) is heading north at 20.0 m/s when it collides with a truck (m_{truck} = 2000 kg) heading east at 15.0 m/s. During the collision the bumpers lock, holding the car and truck together. Determine the velocity of the pair after the collision.



$$P_{Final-x} = m_{Truck}v_{Truck} = (2000\ kg)\left(15.0\frac{m}{s}\right)$$
$$= 30{,}000\ kg\cdot\frac{m}{s}$$

$$P_{Final-y} = m_{car}v_{car} = (1300\ kg)\left(20.0\frac{m}{s}\right)$$
$$= 26{,}000\ kg\cdot\frac{m}{s}$$

$$P_{Final} = \sqrt{P_{Final-x}^2 + P_{Final-y}^2} = \sqrt{\left(30{,}000\ kg\cdot\frac{m}{s}\right)^2 + \left(26{,}000\ kg\cdot\frac{m}{s}\right)^2} = 39{,}699\ kg\cdot\frac{m}{s}$$

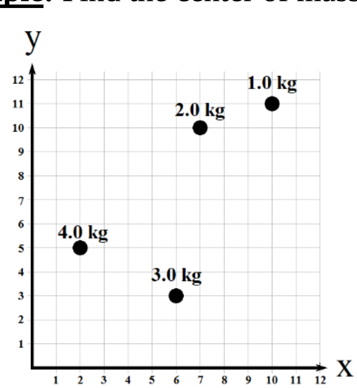
$$v_{Final} = \frac{P_{Final}}{m} = \frac{39{,}699\ kg\cdot\frac{m}{s}}{(2000\ kg + 1300\ kg)} = 12.0\frac{m}{s}$$

Center of Mass

- The dynamics of any object are equivalent to having the entire mass at a single point, the center of mass.
  - This allows us to treat every object as a point with mass.
  - As this is also true for gravity, the center of mass is also called the center of gravity.
- The Center of Mass of an object is the mean (average) position of its mass.
  - For scattered point masses:

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} \qquad y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_iy_i}{\sum_{i=1}^{n} m_i}$$

Example: Find the center of mass.



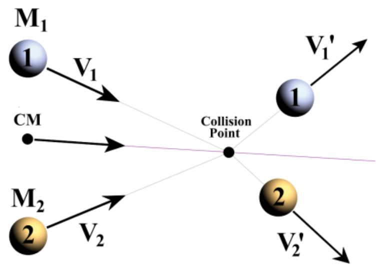
$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$x_{CM} = \frac{(4.0\ kg)(2.0\ m) + (3.0\ kg)(6.0\ m) + (2.0\ kg)(7.0\ m) + (1.0\ kg)(10.0\ m)}{4.0\ kg + 3.0\ kg + 2.0\ kg + 1.0\ kg} = 5.0\ m$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$y_{CM} = \frac{(4.0\ kg)(5.0\ m) + (3.0\ kg)(3.0\ m) + (2.0\ kg)(10.0\ m) + (1.0\ kg)(11.0\ m)}{4.0\ kg + 3.0\ kg + 2.0\ kg + 1.0\ kg} = 6.0\ m$$

- The center of mass of a system of particles of net mass M, moves like a particle of net mass M. When subject to the next external forces it accelerates according to $F_{Net} = Ma$.



$$v_{cm} = \frac{dx_{cm}}{dt} = \frac{d}{dt}\left(\frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}\right)$$

$$v_{cm} = \frac{d}{dt}\left(\frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{M}\right)$$

$$v_{cm} = \frac{m_1}{M}\frac{dx_1}{dt} + \frac{m_2}{M}\frac{dx_2}{dt} + \frac{m_3}{M}\frac{dx_3}{dt} + \cdots$$

$$v_{cm} = \frac{m_1v_1}{M} + \frac{m_2v_2}{M} + \frac{m_3v_3}{M} + \cdots = \frac{m_1v_1{}'}{M} + \frac{m_2v_2{}'}{M} + \frac{m_3v_3{}'}{M} + \cdots$$

- For mass distributions:

$$x_{CM} = \frac{\int x dm}{\int dm} \qquad y_{CM} = \frac{\int y dm}{\int dm}$$

- The object is broken into infinitesimally small pieces where ‘dm’ is the mass of any given piece. These are then added together (integration).
- Typically, dm is written in terms of volume and density.

$$m = \rho \cdot V \quad \text{therefore…} \quad dm = \rho \cdot dV$$

Example: Find the x-component of the center of mass of the triangle shown. Assume uniform thickness and density.

Step 1: Split mass into small pieces, each with the same value of x (as x appears in our equation)

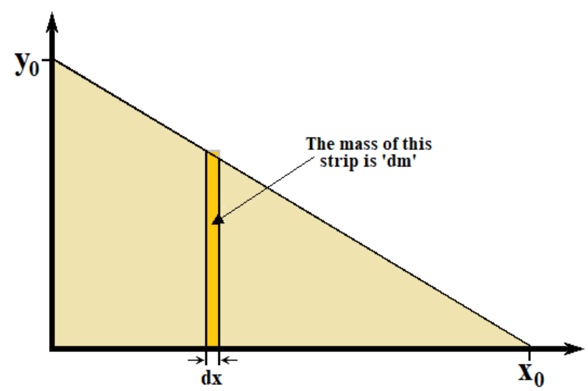
Step 2: Find ‘dm’, the mass of the strip.

$$dm = \rho dV = \rho z_0 A = \rho z_0 y dx$$

In this instance, we must find y as a function of x (since y varies with x).

$$y = mx + b \qquad m = \frac{\Delta y}{\Delta x} = \frac{0 - y_0}{x_0 - 0} = -\frac{y_0}{x_0}$$

$$b = y_0 \qquad dm = \rho z_0 y dx = \rho z_0 (mx + b)dx$$



Step 3: Plug in and integrate.

$$x_{CM} = \frac{\int x dm}{\int dm} = \frac{\int_0^{x_0} x[\rho z_0(mx+b)dx]}{\int_0^{x_0} \rho z_0(mx+b)dx} = \frac{\rho z_0 \int_0^{x_0}(mx^2+bx)dx}{\rho z_0 \int_0^{x_0}(mx+b)dx} = \frac{\int_0^{x_0}(mx^2+bx)dx}{\int_0^{x_0}(mx+b)dx}$$

$$x_{CM} = \frac{\left[\frac{1}{3}mx^3 + \frac{1}{2}bx^2\right]_0^{x_0}}{\left[\frac{1}{2}mx^2 + bx\right]_0^{x_0}} = \frac{\frac{1}{3}mx_0^3 + \frac{1}{2}bx_0^2}{\frac{1}{2}mx_0^2 + bx_0} = \frac{2mx_0^3 + 3bx_0^2}{3mx_0^2 + 6bx_0} = \frac{2\left(-\frac{y_0}{x_0}\right)x_0^3 + 3y_0x_0^2}{3\left(-\frac{y_0}{x_0}\right)x_0^2 + 6y_0x_0}$$

$$x_{CM} = \frac{-2y_0x_0^2 + 3y_0x_0^2}{-3y_0x_0 + 6y_0x_0} = \frac{y_0x_0^2}{3y_0x_0} = \frac{1}{3}x_0$$